Correction Model Midterm Exam 24/3/17

1. a) $\{1, t\}$ is a basis of S. It turns out that 1 and t are orthogonal:

$$(1, t) = \int_{-1}^{1} t\, dt = 0$$

We normalize 1 and t to turn them into orthonormal basis vectors:

$$\|t\|^2 = \int_{-1}^{1} t^2\, dt = \frac{1}{3} t^3 \Big|_{-1}^{1} = \frac{2}{3}$$

$$\Rightarrow \|t\| = \sqrt{\frac{2}{3}}$$

$$\|1\|^2 = \int_{-1}^{1} dt = 2 \Rightarrow \|1\| = \sqrt{2}$$

Required orthormal basis: $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, t \right\}$

b) Let $p(t)$ be the best least squares approximation of $f(t) = e^t$. Then $p(t)$ is the projection onto S, given by

$$p(t) = \left(e^t, \frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} + \left(e^t, \sqrt{\frac{3}{2}}\, t\right) \cdot \sqrt{\frac{3}{2}}\, t$$

$$= \frac{1}{2}(e^t, 1) + \frac{3}{2}(e^t, t) \cdot t$$

We compute: $(e^t, 1) = \int_{-1}^{1} e^t\, dt = e - \frac{1}{e}$

and $(e^t, t) = \int_{-1}^{1} t e^t\, dt = \int_{-1}^{1} t\, d e^t =$

$$t e^t \Big|_{-1}^{1} - \int_{-1}^{1} e^t\, dt = t e^t \Big|_{-1}^{1} - e^t \Big|_{-1}^{1} =$$

$$e + \frac{1}{e} - e + \frac{1}{e} = \frac{2}{e}$$

Thus $p(t) = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{2}\,\frac{2}{e}\cdot t$

$= \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e}t$

2. a) Let S be such that $S^{-1}AS = \Lambda$ with Λ diagonal. Clearly, $S^{-1}.I.S = I$ is also diagonal.

b) Let S be such that $S^{-1}AS = \Lambda_1$ and $S^{-1}BS = \Lambda_2$ with Λ_1 and Λ_2 diagonal. Then $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Thus we get

$$\begin{aligned} AB &= S\Lambda_1 S^{-1} S \Lambda_2 S^{-1} \\ &= S\Lambda_1\Lambda_2 S^{-1} \\ &= S\Lambda_2\Lambda_1 S^{-1} \\ &= S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = BA \end{aligned}$$

c) Let $A = (a_{ij})$ be an $n \times n$ matrix. Let

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

with $d_i \neq d_j$ for $i \neq j$. Assume $DA = AD$. This yields:

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} d_1 a_{11} & d_1 a_{12} & \cdots & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & \cdots & d_2 a_{2n} \\ \vdots & & & \\ d_n a_{n1} & d_n a_{n2} & \cdots & d_n a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} = \begin{pmatrix} d_1 a_{11} & d_2 a_{12} & \cdots & d_n a_{1n} \\ d_1 a_{21} & d_2 a_{22} & \cdots & d_n a_{2n} \\ \vdots & & & \vdots \\ d_1 a_{n1} & d_2 a_{n2} & & d_n a_{nn} \end{pmatrix}$$

Hence we read off that for all $i \neq j$

$$d_i a_{ij} = d_j a_{ij}$$

Since $d_i \neq d_j$ this implies $a_{ij} = 0$ for $i \neq j$
Conclusion: A is a <u>diagonal matrix</u>

d) Assume $AB = BA$. Since A has distinct eigenvalues there exists S such that $S^{-1}AS = \Lambda$ with Λ diagonal.
I claim now that S also diagonalizes B, i.e. $S^{-1}BS = C$ is diagonal.

First note that Λ has distinct entries on the diagonal. I will now show that $C\Lambda = \Lambda C$:

$$\begin{aligned} C\Lambda &= S^{-1}BSS^{-1}AS \\ &= S^{-1}BAS \\ &= S^{-1}ABS \\ &= S^{-1}ASS^{-1}BS = \Lambda C \end{aligned}$$

By problem (c) we conclude that C is diagonal.

3. a) Write the equation in the following way:

$$A(x_1\, x_2 \dots x_n) - (x_1 \dots x_n)\begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} = (c_1\, c_2 \dots c_n)$$

This is equivalent with

$$(Ax_1\; Ax_2 \dots Ax_n) - (b_1x_1\; b_2x_2 \dots b_nx_n)$$
$$= (c_1\; c_2 \dots c_n)$$

Thus $X = (x_1 \dots x_n)$ is a solution of the Sylvester equation if and only if for $i = 1, 2, \dots, n$ we have

$$Ax_i - b_i x_i = c_i$$

b) If b_i is not an eigenvalue of A then $A - b_i I$ is nonsingular. Rewrite

$$Ax_i - b_i x_i = c_i$$

as

$$(A - b_i I)x_i = c_i$$

This yields $x_i = (A - b_i I)^{-1} c_i$. Doing this for all $i = 1, 2, \dots n$ we find $X = (x_1, x_2 \dots x_n)$ as a solution.

c) If $AX - XB = C$ then $AXU^T - XBU^T = CU^T$ so $AXU^T - XU^TUBU^T = CU^T$. Define $Y := XU^T$. Then

$$AY - YUBU^T = CU^T$$

Conversely, if $AY - YUBU^T = CU^T$ has a solution Y then $AYU - YUB = C$ so $X := YU$ satisfies $AX - XB = C$

d) Let U be a unitary matrix such that $UBU^T = \Lambda$ is diagonal. Then we get

$$AY - Y\Lambda = CU^T$$

If A and B have no common eigenvalues then the diagonal elements of Λ are not eigenvalues of A. By part (b), there is then a solution Y. By part (c) the original equation has a solution X.

4. (a) $A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. This gives that A has singular values $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$. We compute now the eigenvectors of $A^T A$ corresponding to 3 and 1

$$A^T A v_1 = 3 v_1 \quad \text{yields} \quad v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A^T A v_2 = v_2 \quad \text{yields} \quad v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Thus, our "V_matrix" is

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

We now compute the "U_matrix". We must have

$$AV = U\Sigma$$

or

$$A(v_1 \; v_2) = (u_1 \; u_2 \; u_3)\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Hence $u_1 = \frac{1}{\sqrt{3}} A v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$

$$u_2 = A v_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

We compute u_3 as any unit vector orthogonal to $\mathrm{span}(u_1, u_2)$. For example

$$u_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

So $U = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

b) The best rank 1 approximation is

$$U \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} V^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$